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On Extended Versions of Dancs-Hegedüs-Medvegyev's Fixed Point Theorem

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In this article we establish some fixed point (known also as critical point, invariant point) theorems in *quasi-metric spaces*. Our results unify and further extend in some regards the fixed point theorem proposed by Dancs et al. (1983), the results given by Khanh and Quy (2010, 2011), the preorder principles established by Qiu (2014), and the results obtained by Bao et al. (2015). In addition, we provide examples to illustrate that the improvements of our results are significant.

Keywords: Ekeland Variational Principle, Fixed Point, Quasi Metric, Forward Cauchy Sequence, Forward Convergence.

AMS Subject Classification: 49J53, 49J52 47J30, 54H25, 90C29, 90C30.

1. Introduction

The celebrated Ekeland variational principle has been recognized as a fundamental tool in the study of various aspects of optimization theory and variational analysis. Since it has been established, it has found many applications in different fields in Analysis. For instance, it has been used to prove the infinite-dimensional mountain path theorem of Ambrosetti and Rabinowitz [1] and has been the key ingredient for proving new variational principles such as the Borwein-Preiss variational principle [2]. It has provided simple and elegant proofs of known results such as the Caristi fixed point theorem in complete metric spaces [3] (in fact the two results are equivalent). It is well established that Dancs-Hegedüs-Medvegyev's fixed point theorem [4, Theorem 3.1] has served as a significant tool in proving Ekeland's variational principle [5] and its extensions to vector and set optimization; the reader is referred for instance to [6–13]. It is important to emphasize that the Dancs-Hegedüs-Medvegyev fixed point theorem is equivalent to Ekeland's variational principle [5] in the sense that one implies the other. The plan of the paper is organized as follows. We begin in section 2 with recalling the Dancs-Hegedüs-Medvegyev fixed point theorem and some of its recent developments. Through this section we recall also some concepts and notations that we will use in the rest of the paper. Our work requires the concept of quasi-metric space, which we review in section 3. Armed with the previous results and such quasi-metric tools, in section 4, we establish in Theorem 4.1 an unified version of Dancs-Hegedüs-Medvegyev

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fixed point theorem, as well as an "all sequences" version of Theorem 4.1 .Finally, remarks on further research topics are given in section 5.

2. Some recent developments: a brief survey

For convenience of the reader, let us recall the Dancs-Hegedüs-Medvegyev fixed point theorem and some of its recent developments. Throughout, we will use the notation"$\Phi : X \rightrightarrows X$" to denote a set-valued mapping, that is a mapping assigning to each point $x \in X$, a subset (possibly empty) $\Phi(x)$ of X and we say that $\{x_n\} \subset X$ is a generalized Picard sequence of Φ, if $x_{n+1} \in \Phi(x_n)$ for all $n \in \mathbb{N}$.

THEOREM 2.1 **([4, Theorem 3.1]).** *Let (X, d) be a complete metric space, and let $\Phi : X \rightrightarrows X$ be a set-valued mapping satisfying the following conditions:*

(A1) *$\Phi(x)$ is a closed set for all $x \in X$;*
(A2) *$x \in \Phi(x)$ for all $x \in X$;*
(A3) *$x_2 \in \Phi(x_1) \Longrightarrow \Phi(x_2) \subset \Phi(x_1)$ for all $x_1, x_2 \in X$;*
(A4) *For each generalized Picard sequence $\{x_n\} \subset X$ of Φ, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.*

Then, for every starting point $x_0 \in X$, there is a convergent sequence $\{x_n\} \subset X$ whose limit x_* is a fixed point of Φ, i.e., $\Phi(x_*) = \{x_*\}$.

In [10, 11], Khanh and Quy presented an extension of Theorem 2.1 in order to establish a new version of Ekeland's variational principle for weak τ-functions.

Definition 1 **(τ-functions and weak τ-functions [10, 11]).** Let (X, d) be a metric space. A bifunction $p : X \times X \to \mathbb{R}_+$ is called a *τ-function* whenever the following four conditions hold:

(τ1) $p(x, z) \le p(x, y) + p(y, z)$ (triangle inequality);
(τ2) for all $x \in X$, $p(x, \cdot)$ is lower semicontinuous (lower semicontinuity);
(τ3) for all sequences $\{x_n\}$, $\{y_n\}$ with $\lim_{n\to\infty} p(x_n, y_n) = 0$ and $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$, one has $\lim_{n\to\infty} d(x_n, y_n) = 0$ (p-convergence implies d-convergence);
(τ4) $p(x, y) = 0$ and $p(x, z) = 0$ imply that $y = z$ (indistancy implies coincidence).

A bifunction $p : X \times X \to \mathbb{R}_+$ is called *weak τ-function* whenever it satisfies conditions (τ1), (τ3), and (τ4).

THEOREM 2.2 **([11, Lemma 3.4])** *Let (X, d) be a metric space, p be a weak τ-function on X, and $\Phi : X \rightrightarrows X$ be a set-valued mapping. Suppose given a generalized Picard sequence $\{x_n\} \subset X$ of Φ convergent to $\bar{x}$ with respect to p in the sense that $\lim_{n\to\infty} p(x_n, \bar{x}) = 0$ with the following properties:*

(B1) *$\Phi(x_{n+1}) \subset \Phi(x_n)$ for all $n \in \mathbb{N}$;*
(B2) *$\lim_{n\to\infty} \sup_{u\in\Phi(x_n)} p(x_n, u) = 0$;*
(B3) *$\bar{x} \in \Phi(x_n)$ for all $n \in \mathbb{N}$.*

Then,

$$\bigcap_{n\in\mathbb{N}} \Phi(x_n) = \{\bar{x}\}.$$

Assume, in addition, that

(B4) $\Phi(\bar{x}) \neq \emptyset$ *and* $\Phi(\bar{x}) \subset \Phi(x_n)$ *for all* $n \in \mathbb{N}$.

Then $\bar{x}$ *is an invariant point of* Φ, *i.e.,* $\Phi(\bar{x}) = \{\bar{x}\}$.

Remark 1 In [11, Lemma 3.4], Khanh and Quy imposed assumptions on one generalized Picard sequence under consideration instead of all sequences in the original or similar results. It is important to emphasize that Dancs et al.'s proof in [4, Theorem 3.1] also holds under the validity of conditions (B1)–(B4). In fact, conditions (A1)–(A4) ensure the existence of a generalized Picard sequence of Φ which satisfies condition (B1)–(B4). It is worth emphasizing that the proof of [11, Lemma 3.4] is nothing but the middle part of Khanh-Quy's version of Ekeland's variational principle.

In [12], Qiu established a general preorder principle from which most of the known set-valued Ekeland variational principles and their improvements were derived. However, it could not imply Khanh and Quy's EVP in the afore-mentioned papers [11] in which a weak τ-function plays the role of the metric in the original principle since the generalized distance between two distinct points x and y of a weak τ-function $p(x, y)$ may be zero. Then, Qiu further revised it to a more general version in [13, Theorem 2.1].

Definition 2 **(preordered and ordered sets)** Let Ξ be a nonempty set and $Q \subset \Xi \times \Xi$ be a subset of the cartesian product $\Xi \times \Xi$. Let us define a binary relation $\preceq$ associated to Q on Ξ by

$$v \preceq z \;:\Longleftrightarrow\; (v, z) \in Q.$$

The binary relation $\preceq$ is a *preorder*; known also as a *quasiorder*, whenever it satisfies the following properties:

$\big[\, \forall\, z \in \Xi,\; z \preceq z \,\big]$ (reflexivity) and
$\big[\, \forall\, z, z', z'' \in \Xi,\; z \preceq z' \;\wedge\; z' \preceq z'' \implies z \preceq z'' \,\big]$ (transitivity).

A set equipped with a preorder is called a *preordered* set. When $\Xi = Z$ is a vector space, we call the pair $(Z, \preceq)$ a preordered vector space. If a preorder is also *antisymmetric*, i.e.,

— $z \preceq v \;\wedge\; v \preceq z \implies v = z$ (antisymmetry),

then it is a *partial order*.

THEOREM 2.3 **([13, Theorem 2.1])** *Let* $(X, \preceq)$ *be a preordered set and consider the level-set mapping* $S : X \rightrightarrows X$ *of the set* X *with respect to the preorder* $\preceq$ *defined by*

$$S(x) := \{u \in X |\; u \preceq x\}. \tag{2.1}$$

Let $x_0 \in X$ *be such that* $S(x_0) \neq \emptyset$ *and consider* $\varphi : (X, \preceq) \to \mathbb{R} \cup \{\pm\infty\}$ *an extended real-valued function which is monotone with respect to* $\preceq$. *Suppose that*

(C1) $-\infty < \inf\{\varphi(x) |\; x \in S(x_0)\} < +\infty$;

(C2) *For any* $x \in S(x_0)$ *with* $-\infty < \varphi(x) < +\infty$ *and for any* $z_1, z_2 \in S(x)$ *with* $z_1 \neq z_2$, *one has* $\varphi(x) > \min\{\varphi(z_1), \varphi(z_2)\}$;

(C3) *For any generalized Picard sequence $\{x_n\} \subset S(x_0)$ of S satisfying*

$$\varphi(x_n) - \inf_{x \in S(x_{n-1})} \varphi(x) \to 0 \quad \text{as} \quad n \to \infty,$$

there exists $x_* \in X$ such that $x_* \in S(x_n)$ for all $n \in \mathbb{N}$.

Then, there exists $\bar{x} \in X$ such that

(i) $\bar{x} \in S(x_0)$;
(ii) $S(\bar{x}) \subset \{\bar{x}\}$ *which holds as equality provided that* $S(\bar{x}) \neq \emptyset$.

Remark 2 Since $\preceq$ is a preorder, the level-set mapping S defined by (2.1) automatically satisfies conditions (A2) and (A3). It plays the role of the set-valued mapping Φ in Theorems 2.1 and 2.2. The function φ is nothing but a utility function associated to the preorder $\preceq$. It allows the author not to impose some topological properties on X. Later, we will show that some relaxation of the completeness and separation properties of the metric space (X, d) in Dansc et al.'s result are fulfilled under the imposed assumptions (C1)–(C3).

In [6, 7], Bao, Mordukhovich and Soubeyran established a far going extension of Dancs-Hegedüs-Medvegyev's fixed point theorem for parametric multifunctions in quasi-metric spaces. This extension can also be interpreted as an existence theorem of minimal points with respect to reflexive and transitive preferences for sets in products spaces with possible applications to behavioral sciences. Below is a simple version of [6, Theorem 2.3].

3. Quasi-metric spaces: definition, basic properties and examples

Definition 3 **(quasi-metric spaces)** A quasi-metric space (also called quasi-pseudo-metric space by Reilly et al. [14]) is a pair (X, q) consisting of a set X and a function $q : X \times X \longmapsto \mathbb{R}_+ := [0, \infty)$ on $X \times X$ having the following three properties:

(i) $q(x, x') \geq 0$ for all $x, x' \in X$ and $q(x, x) = 0$ for all $x \in X$ (positivity);
(ii) $q(x, x'') \leq q(x, x') + q(x', x'')$ for all $x, x', x'' \in X$ (triangle inequality).

There is an abundant literature devoted to "distances" where the requirement of symmetry is omitted. Quasi-metrics are common in real life. For example, given a set X of mountain villages, the typical walking times between elements of X form a quasi-metric because traveling up hill takes longer than traveling down hill. Another example is a geometry topology having one-way streets, where a path from point A to point B comprises a different set of streets than a path from B to A. These "metrics" have some interest in topology, but they are also used in applied mathematics in the calculus of variation. Recently, Bao et al. studied in [8, 15, 16] some mathematical models arising in some areas of behavioral sciences (called sometimes "theories of stability/stay and change"). It seems that everyone agrees that the cost to change in these models does not satisfy the symmetry property. Note that several terminologies are used for the concept of what we call in this paper, quasi-metric: Mennuci [17] uses the term *asymmetric semidistance*, Cobzaş [18] and Reilly et al. [14] speak about *quasi-pseudo-metrics*, while Deza et al. [19], use the name *quasi semi-metric* and Mainik and Mielke [20] employ the name of dissipation distance.

As well-known, if in addition, a quasi-metric satisfies the *symmetry* property

$q(x,x') = q(x',x)$ for all $x,x' \in X$, then q is a *metric*. Part (ii) in the previous definition of a quasi-metric was formalized by Hausdorff in the celebrated monography "*Grundzüge der Mengenlehre*" [21, p. 145–146] which is considered as the foundation of the theory of topological and metric spaces (see details in [19] and [22]). Part (ii) was first formalized by Fréchet [23] and later treated by Hausdorff [21]. Some examples of quasi-metrics are listed below:

- the Sorgenfrey quasi-metric on $\mathbb{R}$, defined by $q(x,y) = y - x$ if $y \geq x$ and $q(x,y) = 1$ otherwise. This quasi-metric describes the process of filing down a metal stick: it is easy to reduce its size, but it is difficult or impossible to grow it;
- the quasi-metric on $\mathbb{R}$ defined by $q(x,y) = \max(y-x, 0)$;
- the real half-line quasi-semi-metric defined by $q(x,y) = \max(0, \ell n \frac{y}{x})$ on the set of strictly positive reals;
- the circular-railroad distance, see, [24, Example 2.2]. Imagine a circular railroad line which moves only in a counterclockwise direction around a circular track, represented by the unit circle $\mathcal{S}^1$. The circular-railroad quasi-metric from any point, $x \in \mathcal{S}^1$, to any other point, $y \in \mathcal{S}^1$, is simply the counterclockwise circular arc length from x to y in $\mathcal{S}^1$;
- the dissipation distance related to the energetic formulation of energetic models for rate-independent systems [20]: consider $X := \{u \in L^1(\Omega, \mathbb{R}^p) : \|u\|_\infty \leq 1\}$ equipped with the weak L^1- topology and the dissipation distance defined by $q(u_1,u_2) = \|u_1 - u_2\|_{L^1}$.
- the Minkowski gauge function defined on $\mathbb{R}^n$ by $q_B(x,y) = \inf\{\alpha > 0 : y - x \in \alpha B\}$, where B is a convex compact subset of $\mathbb{R}^n$.

Since the conjugate bifunction $\overline{q} : X \times X \to \mathbb{R}_+$ of a quasi-metric defined by $\overline{q}(x,y) = q(y,x)$ is also a quasi-metric. Following Kelly [25], the space $(X, q, \overline{q})$ is called a bitopological spaces with two topologies:

- the topology τ_q generated by the balls with center $x \in X$ and radius ε and defined by $\mathbb{B}_q(x;r) := \{y \in X : q(x,y) < \varepsilon\}$;
- the topology $\tau_{\overline{q}}$ generated by the balls with center $x \in X$ and radius ε and defined by $\mathbb{B}_{\overline{q}}(x;r) := \{y \in X : \overline{q}(x,y) < \varepsilon\} = \{y \in X : q(y,x) < \varepsilon\}$.

The balls $\mathbb{B}_q(x;r)$ and $\mathbb{B}_{\overline{q}}(x;r)$ are called forward and backward balls by Menucci [17] and left and right balls by Cabzaş [18]. These two topologies allow us to define two notions of convergences associated to the quasi-metric q:

Definition 4 **(convergences in quasi-metric spaces).**

(i) A sequence $\{x_n\}$ is said to be backward convergent to x_∞, if it is convergent with respect to the topology τ_q, i.e., $\lim\limits_{n\to+\infty} q(x_\infty, x_n) = 0$.

(ii) A sequence $\{x_n\}$ is said to be forward convergent to x_∞, if it is convergent with respect to the topology $\tau_{\overline{q}}$, i.e., $\lim_{n\to+\infty} q(x_n, x_\infty) = 0$.

Since a quasi-metric may fail to be symmetric, the quasi-distances $q(x_n, x_m)$ and $q(x_m, x_n)$ are different. The definition of Cauchy sequences in metric spaces takes two following forms.

Definition 5 **(Cauchy sequences in quasi-metric spaces).**

(i) A sequence $\{x_n\}$ is said to be forward Cauchy if for every $\varepsilon > 0$, there is some $N_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$ and every $k \in \mathbb{N}$, then $q(x_n, x_{n+k}) < \varepsilon$.

(ii) A sequence $\{x_n\}$ is said to be backward Cauchy if for every $\varepsilon > 0$, there is some $N_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$ and and every $k \in \mathbb{N}$, then $q(x_{n+k}, x_n) < \varepsilon$.

Note that in a metric space the two concepts coincide with the usual concept of a Cauchy-sequence.

Remark 3 At this point, it is important for the reader to to be aware of the differences bettwen our definitions and the ones previously used. The backward convergence is termed either q-convergence or convergence w.r.t. τ_q in [25, 26]. The forward convergence is called as $\overline{q}$-convergence or convergence w.r.t. the topology τ_r in the aforementioned references, and left sequential convergence in [6–8, 15] and many references therein. The forward Cauchy sequence is known as left-sequential Cauchy (also as lelf Cauchy or Cauchy) sequence in Bao et al. [6–8], left-K-Cauchy in Reilly et al. [26]. The backward Cauchy notion is known as p-Cauchy in [25, Definition 2.10], right-K-Cauchy in Reilly et al. [26]. The forward completeness is used in [6–8, 15] as left-sequential completeness and in Reilly [26] as left-K-completness. The backward completeness was studied in Kelly [25] under the name of p-completness and in Reilly [26] as right-K-completness.

This change of notation is motivated by the following consideration (private communication with A. Soubeyran): denoting the state of an object at the time n by x_n, then the future (forward) state is x_{n+1}. Then, $q(x_n, x_{n+1})$ is the cost to change from x_n to x_{n+1}. If the expected /ideal state is x_∞, then the cost to change from the current state to the ideal state is$q(x_n, x_\infty)$ which should be called a forward cost.

According to Reilly et al., [14, Example 1. p. 130], a sequence could be forward convergent without being backward convergent and a sequence could be convergent without being forward or backward convergent.

Definition 6 **(completeness in quasi-metric spaces).**

- **(i)** The space (X, q) is forward (resp. backward) Hausdorff, if every forward (resp. backward) converging sequence has a unique forward (resp. backward) limit point.
- **(ii)** The space (X, q) is forward (resp. backward) complete, if every forward (resp. backward) Cauchy sequence is forward (resp. backward) convergent.
- **(iiii)** The space (X, q) is forward-backward (resp. backward-forward) complete, if every forward (resp. backward) Cauchy sequence is backward (resp. forward) convergent.

THEOREM 3.1 **([7, Corollary 4.5])** *Let (X, q) be a forward complete and forward Hausdorff* [1] *quasi-metric space, and let $\Phi : X \rightrightarrows X$ be a set-valued mapping satisfying the conditions:*

- **(D1)** $x \in \Phi(x)$ *for all* $x \in X$*;*
- **(D2)** $u \in \Phi(x) \Longrightarrow \Phi(u) \subset \Phi(x)$ *for all* $x, u \in X$*;*
- **(D3)** *For each generalized Picard sequence* $\{x_n\}$ *of* Φ *being forward convergent* [2]*to* x_**, then* $x_* \in \Phi(x_n)$ *for all* $n \in \mathbb{N}$*;*
- **(D4)** *For each generalized Picard sequence* $\{x_n\} \subset X$*,* $\lim\limits_{n \to +\infty} q(x_n, x_{n+1}) = 0$.

Then, for every point $x_0 \in X$ there is a generalized Picard sequence $\{x_n\} \subset X$ of Φ starting from x_0 and forward converging [3]*to an invariant point $\bar{x}$ of Φ, i.e.,*

[1,2,3] the adjective 'forward' (i.e., 'left-sequential' in the original version) was obmitted for simplicity.

$\Phi(\bar{x}) = \{\bar{x}\}$.

In this paper, we establish an unified version for the afore-mentioned results. It takes the 'one sequence' form of Theorem 2.2 in the setting of Theorem 3.1.

4. Main Results

THEOREM 4.1 **(a unified version of DHM's fixed point theorem).** *Let (X, q) be a quasi-metric space, $\Phi : X \rightrightarrows X$ be a set-valued mapping, and $\{x_k\} \subset X$ be a generalized Picard sequence of Φ, i.e., $x_{n+1} \in \Phi(x_n)$ for all $n \in \mathbb{N}$. Assume that the following conditions hold:*

(E1) $\Phi(x_{n+1}) \subset \Phi(x_n)$ *for all* $n \in \mathbb{N}$*;*
(E2) $\lim\limits_{n\to\infty} \sup\limits_{x\in\Phi(x_n)} q(x_n, x) = 0$*;*
(E3) *there is some* $\bar{x} \in X$ *such that* $\bar{x} \in \Phi(x_n)$ *for all* $n \in \mathbb{N}$*;*
(E4) $\{x_n\}$ *has at most one forward limit;*

Then,

$$\bigcap_{n\in\mathbb{N}} \Phi(x_n) = \{\overline{x}\} \tag{4.2}$$

where $\overline{x}$ *is taken from* (E3)*. Assume, in addition, that*

(E5) $\Phi(\overline{x}) \subset \Phi(x_n)$ *for all* $n \in \mathbb{N}$.

Then, $\bar{x}$ *is a nonvariant point of* Φ*, i.e.,* $\Phi(\bar{x}) \subset \{\bar{x}\}$*; it becomes an invariant point of* Φ *provided that* $\Phi(\overline{x}) \neq \emptyset$.

Proof. Suppose given a sequence $\{x_n\} \subset X$ satisfying conditions (E1)–(E4). Obviously, condition (E3) says that

$$\{\overline{x}\} \subset \bigcap_{n\in\mathbb{N}} \Phi(x_n). \tag{4.3}$$

Next, we will prove that the intersection is a singleton. Assume, in addition to $\overline{x}$, that an element $\underline{x}$ also belongs to the left-hand side of (4.3). By condition (E2), we have $\lim\limits_{n\to\infty} q(x_n, \overline{x}) = \lim\limits_{n\to\infty} q(x_n, \underline{x}) = 0$ which ensures that $\underline{x} = \overline{x}$ due to condition (E4) and thus (4.3) holds as an equality, i.e., the common point condition (4.2) holds. Employing now condition (E5) to (4.2) we obtain

$$\Phi(\bar{x}) \subset \bigcap_{n\in\mathbb{N}} \Phi(x_n) = \{\bar{x}\}.$$

The proof is complete. □

PROPOSITION 4.2 *The fulfilment of* (E1)–(E2) *implies that the sequence* $\{x_n\}$ *is a forward Cauchy sequence with respect to the quasi-metric* q *in* X.

Proof. Condition (E2) tells us that for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sup_{u\in\Phi(x_n)} q(x_n, u) < \varepsilon \quad \text{whenever} \quad n \geq N_\varepsilon.$$

Picking now any $n \geq m \geq N_\varepsilon$, we have $x_m \in \Phi(x_m) \subset \Phi(x_n)$ due to (E1) and thus

$$q(x_m, x_n) \leq \sup_{u \in \Phi(x_n)} q(x_n, u) < \varepsilon,$$

which verifies that the sequence $\{x_n\}$ is forward Cauchy in the quasi-metric space (X, q). The proof is complete. $\square$

PROPOSITION 4.3 *Assume that conditions* (E1) *and* (E2) *hold. Assume also that the quasi-metric space* (X, q) *is forward complete and the quasi-metric enjoys the condition* $q(x, y) = 0 \Longleftrightarrow x = y$. *Then, condition* (E3$'$)

(E3$'$) *there is a backward limit* $\bar{x} \in X$ *of the sequence* $\{x_n\}$ *such that* $\bar{x} \in \Phi(x_n)$ *for all* $n \in \mathbb{N}$

implies both conditions (E3) *and* (E4) *in Theorem 4.1.*

Proof. Obviously, (E3$'$) $\Longrightarrow$ (E3). Therefore, it remains to prove the implication (E3$'$) $\Longrightarrow$ (E4). By Proposition 4.2, every generalized Picard sequence is forward Cauchy. By the assumed forward completeness property of X, it is forward convergent to some forward limit $\underline{x} \in X$. The fulfillness of (E3$'$) ensures the existence of some backward limit $\bar{x}$ of $\{x_n\}$, i.e., $\lim_{n\to\infty} q(\bar{x}, x_n) = 0$ such that

$$\bar{x} \in \Phi(x_n) \text{ for all } n \in \mathbb{N}.$$

Taking into account condition (E2), $\bar{x}$ is also a forward limit, i.e., $q(x_n, \bar{x}) \to 0$ as $n \to \mathbb{N}$. Taking into account the triangular inequality to estimate the quasi-distance between $\bar{x}$ and $\underline{x}$, we have $q(\bar{x}, \underline{x}) \leq q(\bar{x}, x_n) + q(x_n, \underline{x})$ for all $n \in \mathbb{N}$ and thus $q(\bar{x}, \underline{x}) = 0$. The additional condition imposed on the quasi-metric implies $\bar{x} = \underline{x}$. Therefore, condition (E4) holds. The proof is complete. $\square$

PROPOSITION 4.4 *Assume that conditions* (E1)-(E2) *hold and the quasi-metric space* (X, q) *is forward complete. Then, condition* (E3$''$)

(E3$''$) *there is* $\bar{x} \in X$ *is a forward limit of the sequence* $\{x_n\}$ *such that* $\bar{x} \in \Phi(x_n)$ *for all* $n \in \mathbb{N}$

is equivalent to condition (E3) *in Theorem 4.1.*

Proof. Obviously, (E3$''$) $\Longrightarrow$ (E3). To justify the reverse implication it is sufficient to show that the imposed conditions (E1)-(E3) implies that the element $\bar{x}$ in (E3) is, indeed, a forward limit of $\{x_n\}$. By Proposition 4.2, the sequence $\{x_n\}$ satisfying (E1)-(E2) is forward Cauchy. Fix an element $\bar{x}$ satisfying (E3). Condition (E2) implies that $\lim_{n\to\infty} q(x_n, \bar{x}) = 0$ which clearly verifies that $\bar{x}$ is a forward limit. The proof is complete. $\square$

Next, we derive from Theorem 4.1 an extension of Theorem 3.1; cf. [7, Corollary 4.5] which can be used to further generalize the Ekeland variational principle and its equivalents.

THEOREM 4.5 **(an 'all sequences' version of Theorem 4.1).** *Let* (X, q) *be a quasi-metric space,* $\Phi : X \rightrightarrows X$ *be a set-valued mapping. Assume that*

(F1) *if* $u \in \Phi(x)$, *then* $\Phi(u) \subset \Phi(x)$ *for all* $u, x \in X$;
(F2) *for any generalized Picard sequence* $\{x_n\} \subset X$, *i.e.,* $x_{n+1} \in \Phi(x_n)$ *for all*

$n \in \mathbb{N}$, if

$$\lim_{n\to\infty} \sup_{x\in\Phi(x_n)} q(x_n, x) = 0,$$

then there exists some element $x_* \in X$ such that $x_* \in \Phi(x_n)$ for all $n \in \mathbb{N}$;

(F3) *any forward Cauchy generalized Picard sequence $\{x_n\} \subset X$ has at most one forward limit;*

(F4) *for each generalized Picard sequence $\{x_n\} \subset X$, $\lim_{n\to+\infty} q(x_n, x_{n+1}) = 0$.*

Then, Φ has a nonvariant point $\overline{x}$ in the sense that $\Phi(\bar{x}) \subset \{\bar{x}\}$. If, furthermore, $\Phi(\bar{x}) \neq \emptyset$, then it is an invariant point of Φ, i.e., $\Phi(\bar{x}) = \{\bar{x}\}$.

Proof. Without any loss of generality, we may assume that $\Phi(x) \neq \emptyset$ for all $x \in X$; otherwise, the result is trivial; any element $\bar{x} \in X$ such that $\Phi(\bar{x}) = \emptyset$ is an invariant point of Φ with $\emptyset = \Phi(\bar{x}) \subset \{\bar{x}\}$. By Theorem 4.1, it is sufficent to show the existence of a generalized Picard sequence satisfying

$$\lim_{n\to\infty} \sup_{x\in\Phi(x_n)} q(x_n, x) = 0.$$

Such a sequence can be inductively constructed by starting with an arbitrary element x_0 and then following the iterative process:

$$x_{n+1} \in \Phi(x_n) \text{ with } q(x_n, x_{n+1}) \geq \sup_{x\in\Phi(x_n)} q(x_n; x) - 2^{-n} \text{ for } n = 0, 1, 2, \ldots \tag{4.4}$$

It is clear that the sequence $\{x_n\}$ is well defined and that the convergence condition (F4) tells us that the quasi-distances $q(x_n, x_{n+1})$ tend to zero as $n \to \infty$. Taking into account the inequality in (4.4) ensures that $\sup_{x\in\Phi(x_n)} q(x_n, x) \to 0$ as $n \to \infty$. The proof is complete. □

Remark 4 Theorem 4.5 is an extension of [6, Corollary 4.5] (Theorem 3.1) due to Proposition 2.4 and the fact that forward (left-sequential) Hausdorff property implies the fulfillment of **(F3)**. The following example illustrates the usage of Theorem 2.1:

Let $X = [0, 1]$ and the quasi-metric on X defined by

$$q(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ 1 & \text{if } x < y. \end{cases}$$

It is not dificult to check that this space is forward complete and forward Hausdorff. Given a forward sequence $\{x_n\}$ in X. For n sufficiently large, we have $q(x_n, x_{n+1}) < 1$. The structure of q yields $x_{n+1} < x_n$ and thus the sequence is decreasing eventually. Since it is bounded from below by 0, it converges to a unique number in $X = [0, 1]$.

Next, we will show that the space (X, q) is not forward-backward complete. Consider the sequence $\{x_n\}$ where $x_n = n^{-1}$. Since $q(x_n, x_m) = n^{-1} - m^{-1} < n^{-1}$ for all $m, n \in \mathbb{N}$ with $m > n$, $\{x_n\}$ is obviously a forward Cauchy sequence. Since $q(0, x_n) = 1$ for all $n \in \mathbb{N}$, the sequence $\{x_n\}$ fails to backward converge to 0. Fix now an arbitrary number $\bar{x} \in (0, 1]$. We have $q(\bar{x}, x_n) = \bar{x} - x_n$ for all n sufficiently large $(n > 1/\bar{x})$ and thus $\lim_{n\to\infty} q(\bar{x}, x_n) = \bar{x} > 0$, i.e., $\bar{x}$ is not a backward limit of

$\{x_n\}$. Since the chosen forward Cauchy sequence $\{x_n\}$ has no backward limit, the space is not forward-backward complete.

Consider now a set-valued mapping $\Phi : X \rightrightarrows X$ with images $\Phi(x) = [0, x]$. Obviously, conditions (E1) and (E2) are satisfied by the chosen sequence with $\Phi(x_n) = [0, n^{-1}]$ and $\sup\limits_{x\in\Phi(x_n)} q(x_n, x) = n^{-1}$. Condition (E3) is fulfilled for $\bar{x} = 0$. We now show that 0 is the only forward limit of $\{x_n\}$. Take an arbitrary number $\bar{x} \in (0, 1]$. For any interger $n \in \mathbb{N}$ with $n > 1/\bar{x}$, one has $x_n = n^{-1} < \bar{x}$ and thus $q(x_n, \bar{x}) = 1$ clearly veryfying that $\bar{x}$ is not a forward limit of $\{x_n\}$. Theorem 4.1 ensures that 0 is an invariant point of Φ.

Next, we will derive from Theorem 4.1 Qiu's revised preorder principle in [13].

THEOREM 4.6 *Theorem 4.1* $\Longrightarrow$ *Theorem 2.3.*

Proof. Assume that all the assumptions in Theorem 2.3 hold. We construct a bi-function $q : X \times X \to \mathbb{R}_+$ with

$$q(x, y) := |\, \varphi(x) - \varphi(y) \,| \ \text{ for all } \ x, y \in X.$$

Due to condition (C1) the function φ is finite valued over $S(x_0)$. The pair $(S(x_0), q)$ is a quasi-metric space since $q(x, x) = |\, \varphi(x) - \varphi(x) \,| = 0$ for all $x \in X$ and

$$\begin{aligned} q(x, z) &= |\, \varphi(x) - \varphi(z) \,| = |\, (\varphi(x) - \varphi(z)) + (\varphi(z) - \varphi(y)) \,| \\ &\le |\, \varphi(x) - \varphi(y) \,| + |\, \varphi(x) - \varphi(y) \,| \\ &= q(x, y) + q(y, z) \ \text{ for all } \ x, y, z \in X. \end{aligned}$$

In order to employ Theorem 4.2 we need to show that the level-set mapping satisfies all four conditions (E1)–(E5).

First, let us construct a generalized Picard sequence starting with x_0 satisfying condition (E2) as follow:

$$\begin{cases} \text{If } \ S(x_{n-1}) = \emptyset, \ \text{then STOP;} \\ \text{If } \ S(x_{n-1}) \neq \emptyset, \text{then choose } \ x_n \in S(x_{n-1}) \ \text{ with } \ \varphi(x_n) < \inf\limits_{u\in S(x_{n-1})} \varphi(u) + 2^{-n}. \end{cases}$$

If there exists n such that $S(x_n) = \emptyset$, then we may take $\overline{x} = x_n$ and clearly it satisfies (i) and (ii). If not, we can obtain a sequence $\{x_n\} \subset S(x_0)$ with $x_{n+1} \in S(x_n)$ for all $n \in \mathbb{N}$ such that

$$\varphi(x_n) < \inf_{u\in S(x_{n-1})} \varphi(u) + 2^{-n}.$$

Obviously, $\varphi(x_n) - \inf\limits_{u\in S(x_{n-1})} \varphi(u) \to 0$ as $n \to \infty$.

Fix an arbitrary integer $n \in \mathbb{N}$. We get from $S(x_n) = \{x \in X |\, u \preceq x_n\}$ that for any $x \in S(x_n)$, $\varphi(x) \le \varphi(x_n) \iff \varphi(x_n) - \varphi(x) \le 0$ due to the monotonicity of

φ. Since

$$\begin{aligned}
&\varphi(x_n) - \inf_{x\in S(x_{n-1})} \varphi(x) \\
&= \varphi(x_n) + \sup_{x\in S(x_{n-1})} \big(-\varphi(x)\big) = \sup_{x\in S(x_{n-1})} \big(\varphi(x_n) - \varphi(x)\big) \\
&= \sup_{x\in S(x_{n-1})} \big|\varphi(x_n) - \varphi(x)\big| = \sup_{x\in S(x_{n-1})} q(x_n; S(x_{n-1})),
\end{aligned}$$

passsing to the limit as $n \to \infty$, we derive that

$$\lim_{n\to\infty} \sup_{x\in S(x_{n-1})} q(x_n; S(x_{n-1})) = 0,$$

which establishes that condition (E2) holds for the sequence $\{x_n\}$. Then, condition (C3) ensures the existence of an element $\overline{x}$ satisfying $\overline{x} \in S(x_n)$ for all $n \in \mathbb{N}$, i.e., condition (E3).

By the transitivity of $\preceq$ and the structure of S, conditions (E1) and (E5) hold. We also have $x \in S(x)$ for all $x \in S(x_0)$ due to the reflexivity of $\preceq$.

It remains to check condition (E4). Assume now that $\lim_{n\to\infty} q(x_n, \overline{x}) = \lim_{n\to\infty} q(x_n, \underline{x}) = 0$ for two elements $\overline{x}$ and $\underline{x}$ in X. Fix $n \in \mathbb{N}$, we get from $\overline{x}, \underline{x} \in S(x_k)$ that $\varphi(\overline{x}) \le \varphi(x_n)$ and $\varphi(\underline{x}) \le \varphi(x_n)$ due to the definition of S and the monotonicity of φ. Therefore, we have

$$\begin{aligned}
&\lim_{n\to\infty} q(x_n, \overline{x}) = \lim_{n\to\infty} q(x_n, \underline{x}) = 0 \\
&\overset{def.}{\Longleftrightarrow} \lim_{n\to\infty} \big|\, \varphi(x_n) - \varphi(\overline{x})\big| = \lim_{n\to\infty} \big|\varphi(x_n) - \varphi(\underline{x})\big| = 0 \\
&\Longleftrightarrow \lim_{n\to\infty} \varphi(x_n) - \varphi(\overline{x}) = \lim_{n\to\infty} \varphi(x_n) - \varphi(\underline{x}) = 0
\end{aligned}$$

which clearly implies that $\varphi(\overline{x}) = \varphi(\underline{x})$. By condition (C2), $\overline{x} = \underline{x}$ clearly verifying the fulfilment of condition (E4). Indeed, if $\overline{x} \neq \underline{x}$, then we get from (C2) that $\varphi(\overline{x}) > \min\{\varphi(\overline{x}), \varphi(\underline{x})\}$ contradicting to $\varphi(\overline{x}) \neq \varphi(\underline{x})$.

We have checked that the chosen sequence $\{x_k\}$ satisfies all the assumptions in Theorem 4.1. Employing this result to $\{x_k\}$, we obtain $S(\bar{x}) = \{\bar{x}\}$, i.e., $\bar{x}$ is an invariant point of S. The proof is complete. □

Finally, let us derive also the Khank-Quy's result in [10, Lemma 3.4].

THEOREM 4.7 *Theorem 4.1* $\Longrightarrow$ *Theorem 2.2.*

Proof. Assume that there is a convergent sequence $\{x_n\}$ such that its limit $\overline{x}$ in the metric space (X, d) satisfies all three conditions (B1)–(B3) in Theorem 2.2.

Define a function $q : X \times X \to \mathbb{R}_+$ by

$$q(x, y) := \begin{cases} p(x, y) & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y. \end{cases} \tag{4.5}$$

It is clear that q is a quasi-metric on X since the positivity and triangle inequality properties hold for q defined in (4.5) due to the definition of q and condition $(\tau 1)$ respectively. Observe that (B1) $\Longleftrightarrow$ (E1) and (B3) $\Longrightarrow$ (E3). Observe also that (B2) $\Longrightarrow$ (E2) since $q(x,y) \le p(x,y)$ for all $x, y \in X$ and thus

$$\lim_{n\to\infty} \sup_{u\in\Phi(x_n)} q(x_n,u) \le \lim_{n\to\infty} \sup_{u\in\Phi(x_n)} p(x_n,u) = 0.$$

Observe finally that condition (E4) holds as well. Conditions (B1) and (B2) ensure that

$$\lim_{n\to\infty} \sup_{m>n} p(x_n,x_m) = 0. \tag{4.6}$$

Define a sequence $\{y_n\}$ by $y_n = x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have $\lim_{n\to\infty} p(x_n,y_n) = 0$. This together with (4.6) implies, by $(\tau 3)$, $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$ clearly implying that $\{x_n\}$ is a Cauchy sequence in the metric space (X,d). By Proposition 4.2, the sequence $\{x_n\}$ is forward Cauchy in (X,q). Assume now that it has a forward limit x_* in (X,q) in the sense that $\lim\limits_{n\to\infty} q(x_n,x_*) = 0$. Then, we have $\lim\limits_{n\to\infty} p(x_n,x_*) = 0$ as well. Arguing by contradiction, assume that $\lim\limits_{n\to\infty} p(x_n,x_*) = \gamma > 0$. Taking into account the definition of q, we get the existance of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \equiv x_*$. This and condition (B2) lead to a contradiction:

$$0 < \gamma \le \lim_{n_k\to\infty} \sup_{u\in\Phi(x_{n_k})} p(x_{n_k},u) = \lim_{n\to\infty} \sup_{u\in\Phi(x_n)} p(x_n,u) = 0.$$

Let $y_n = x_*$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} p(x_n,y_n) = 0$. This together with (4.6) yields $\lim\limits_{n\to\infty} d(x_n,x_*) = 0$ by $(\tau 3)$. Since the sequence $\{x_n\}$ is Cauchy in the complete metric space (X,d), the limit x_* is unique. Therefore, condition (E4) is satisfied for the quasi-metric q defined in (4.5).

We have proved that all the assumptions of Theorem 4.1 are satisfied in the quasi-metric (X,q). Therefore, the conclusion of Theorem 2.2 follows from that of Theorem 4.1. The proof is complete. $\square$

5. Conclusions

The main result of this paper, Theorems 4.1, provides an extension of Dancs-Hegedüs-Medvegyev's fixed point theorem which not only unify several recent generalized versions of this theorem but also further extend them to the quasi-metric space setting. This feature allows us to obtain new applications to Ekeland's variational principle and Caristi's fixed point theorem by using our alternative result (Theorem 4.5). Following this way, we plan to extend this research to the setting of λ-spaces and to the setting of complete cone metric spaces introduced by Lin et al. in [27].

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